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# A Result on the Bruhat Order of a Coxeter Group

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Let $W = (W, S)$ be a Coxeter group with $S$ the set of its Coxeter generators. Let $\leqslant$ be the Bruhat order on $W$. That is, we denote $y \leqslant w$ for $y, w \in W$ if there exist reduced expressions $w = s_1 s_2 \ldots s_r$ and $y = s_{i_1} s_{i_2} \ldots s_{i_t}$ with $s_j \in S$, $1 \leqslant j \leqslant r$, and $i_1, i_2, \ldots, i_t$ a subsequence of $1, 2, \ldots, r$. Let $l(x)$ be the length function on $W$. To each $w \in W$, we associate two subsets of $S$:

$$\mathscr{L}(w) = \{s \in S \mid sw < w\} \qquad \text{and} \qquad \mathscr{R}(w) = \{s \in S \mid ws < w\}.$$

Now we can state our main result as follows.

THEOREM 1. *Suppose that $x, y \in W$ and $s \in S$ satisfy the condition $s \notin \mathscr{L}(y) \cup \mathscr{R}(x)$. Then $xy < xsy$.*

The content of this paper is organized as follows. In Section 1, we shall provide a proof of Theorem 1 and also a generalization of this theorem. Then in Section 2, we shall apply Theorem 1 to investigate some properties of the Hecke algebra associated to $W$ and also to verify a conjecture of L. K. Jones [1].

## 1. THE PROOF OF THEOREM 1

Fix a Coxeter group $(W, S)$. The terminology "an expression of $W$" means an element of $W$ being written as a product of elements of $S$. We say an expression $w$ of $W$ is reduced if $w$ is a product of $l(w)$ elements of $S$. Given two expressions $s_1 s_2 \ldots s_k$ and $t_1 t_2 \ldots t_m$ of $W$ with $s_i$, $t_j \in S$, we write $s_1 s_2 \ldots s_k \equiv t_1 t_2 \ldots t_m$ if $k = m$ and $s_i = t_i$ for all $i$, and write $s_1 s_2 \ldots s_k \not\equiv t_1 t_2 \ldots t_m$ if otherwise. We say that two expressions $g$ and $g'$ are congruent if $g \equiv g'$. Thus we have $g \equiv g' \Rightarrow g = g'$. We say $g'$ is a subexpression of an expression $g$ if $g \equiv x g' y$ for some expressions $x$, $y$ of $W$.



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By a Coxeter transformation on an expression $s_1 s_2 \dots s_t$ with $s_i \in S$, we mean that it is one of the following transformations.

(A) If there exist some $a, b \in S$ and $i, j \in \mathbb{Z}$ with $a \neq b$ and $1 \leqslant i < j \leqslant t$ such that

$$s_i s_{i+1} \dots s_j \equiv aba \dots \quad \text{and} \quad o(ab) = j - i + 1,$$

then we define a transformation

$$s_1 s_2 \dots s_t \mapsto s_1 s_2 \dots s_{i-1} \underbrace{(bab \dots)}_{o(ab)\text{ factors}} s_{j+1} \dots s_t,$$

where the notation $o(x)$ stands for the order of the element $x$.

(B) If there exists some $i \in \mathbb{Z}$, $1 \leqslant i < t$, such that $s_i = s_{i+1}$, then we define a transformation

$$s_1 s_2 \dots s_t \mapsto s_1 s_2 \dots s_{i-1} s_{i+2} \dots s_t.$$

(C) For any $i$, $0 \leqslant i \leqslant t$, and $a \in S$, we define a transformation

$$s_1 s_2 \dots s_t \mapsto s_1 s_2 \dots s_i (aa) s_{i+1} \dots s_t.$$

*Remark* 2. Given any two expressions $s_1 s_2 \dots s_k$ and $t_1 t_2 \dots t_m$ of $W$ with $s_i$, $t_j \in S$ and $s_1 s_2 \dots s_k \not\equiv t_1 t_2 \dots t_m$, it is well known that the expression $s_1 s_2 \dots s_k$ can be passed to $t_1 t_2 \dots t_m$ by a succession of Coxeter transformations. In particular, in the case when $t_1 t_2 \dots t_m$ is a reduced expression, $s_1 s_2 \dots s_k$ can be passed to $t_1 t_2 \dots t_m$ by only performing the Coxeter transformations of kinds (A) and (B).

Define a set of triples

$$T = \{(x, s, y) \mid x, y \in W, s \in S, s \notin \mathscr{L}(y) \cup \mathscr{R}(x)\}.$$

For $i \in \mathbb{Z}$, define

$$T_i = \{(x, s, y) \in T \mid l(x) + l(y) + 1 - l(xsy) = i\}.$$

Clearly, if $T_i \neq \varnothing$ then $i \geqslant 0$ and $i \in 2\mathbb{Z}$. Thus we have a decomposition:

$$T = \bigcup_{j \geqslant 0} T_{2j}.$$

If $(x, s, y) \in T_i$ then we define $p(x, s, y) = i$.

Then Theorem 1 can be reformulated as follows.

**Theorem 3.** *If $(x, s, y) \in T$ then $xy < xsy$.*

It is obvious that

LEMMA 4. *If* $(x, s, y) \in T_0$ *then* $xy < xsy$.

Let $g \equiv s_1 s_2 \dots s_k s t_1 t_2 \dots t_m$ be an expression of $W$ with $s_i, s, t_j \in S$ satisfying the following conditions:

(a) Both $x \equiv s_1 s_2 \dots s_k$ and $y \equiv t_1 t_2 \dots t_m$ are reduced expressions.

(b) $(x, s, y) \in T_{2n}$ for some $n > 0$.

Let $g'$ be an expression obtained from the expression $g$ by a Coxeter transformation $f$ of kind $\neq$(C). Then $f$ must have kind (A).

Suppose that $f$ does not involve the factor $s$. Let $s' = s$, $x' \equiv s'_1 s'_2 \dots s'_k$, and $y' \equiv t'_1 t'_2 \dots t'_m$ be such that $g' \equiv x's'y'$. Then $x'$ and $y'$ are also reduced expressions with $x' = x$ and $y' = y$.

Now suppose that $f$ involves the factor $s$. Then we have

$$g \equiv g_1 \underbrace{(aba \dots)}_{r \text{ factors}} g_2 \qquad \text{and} \qquad g' \equiv g_1 \underbrace{(bab \dots)}_{r \text{ factors}} g_2$$

for some subexpressions $g_1, g_2$ of $g$, where $a, b \in S$ satisfy $a \neq b$ and $r = o(ab)$, and $s$ is the $i$th factor in the parentheses of the expression $g$ with $1 \leqslant i \leqslant r$. Then by choosing $s'$ to be the $(r+1-i)$th factor in the parentheses of the expression $g'$, we have the expression

$$g' \equiv g_1(bab \dots) g_2 \equiv s'_1 s'_2 \dots s'_{k'} s' t'_1 t'_2 \dots t'_{m'}$$

with $s'_i, t'_j \in S$ satisfying the following conditions:

(i) $k' = k + r + 1 - 2i$, $m' = m + 2i - r - 1$.

(ii) Let $x' \equiv s'_1 s'_2 \dots s'_{k'}$ and $y' \equiv t'_1 t'_2 \dots t'_{m'}$. Then $x'y' = xy$ and $(x', s', y') \in T_{2h}$ for some $h \leqslant n$. The equality $h = n$ holds if and only if both the expressions $x'$ and $y'$ are reduced.

To sum up, we have the following result.

LEMMA 5. *Let* $g \equiv s_1 s_2 \dots s_k s t_1 t_2 \dots t_m$, $x$, $y$, $n$ *be as above. Let* $g'$ *be an expression obtained from the expression* $g$ *by a Coxeter transformation* $f$ *of kind* $\neq$(C). *Then we can choose a factor* $s'$ *in the expression* $g'$ *such that*

(a) $g' \equiv s'_1 s'_2 \dots s'_{k'} s' t'_1 t'_2 \dots t'_{m'}$ *with* $s'_i, s'_j, t' \in S$.

(b) *Let* $x' \equiv s'_1 s'_2 \dots s'_{k'}$ *and* $y' \equiv t'_1 t'_2 \dots t'_{m'}$. *Then* $x'y' = xy$ *and* $(x', s', y') \in T_{2h}$ *for some* $h \leqslant n$. *The equality* $h = n$ *holds if and only if both the expressions* $x'$ *and* $y'$ *are reduced.*

Again assume that $g \equiv s_1 s_2 \dots s_k s t_1 t_2 \dots t_m$, $x$, $y$, $n$ are as above. If $n > 0$, then by Remark 2, there exists a sequence of expressions $g_0 \equiv g, g_1, ..., g_h$ of $xsy$ for some $h>0$ such that for every $i$, $1 \leqslant i \leqslant h$, $g_i$ is obtained from $g_{i-1}$ by a Coxeter transformation of kind $\neq$(C) and $g_h$ is a reduced expression. By Lemma 5, we see that there must exist some integer $u$ with $1 \leqslant u < h$ such that

(i) $g_i \equiv s(i, 1) ... s(i, k_i)\, s(i)\, t(i, 1) ... t(i, m_i)$ with $s(i, j)$, $s(i)$, $t(i, j) \in S$ for any $i$, $0 \leqslant i < u$.

(ii) The expressions $s(i, 1) ... s(i, k_i)$ and $t(i, 1) ... t(i, m_i)$ are reduced for all $i$, $0 \leqslant i < u$.

(iii) Either $s(u, 1) ... s(u, k_u)$ or $t(u, 1) ... t(u, m_u)$ is not a reduced expression.

(iv) Let $x_i = s(i, 1) ... s(i, k_i)$ and $y_i = t(i, 1) ... t(i, m_i)$ for $0 \leqslant i \leqslant u$. Then $x_i y_i = xy$ and $(x_i, s(i), y_i) \in T_{2n_i}$ with $n_i = n$ for $0 \leqslant i < u$ and $n_u < n$.

Given $(x, s, y) \in T$, we call a sequence of expressions, say $g_0, g_1, ..., g_u$ for some $u \geqslant 0$, of $xsy$ a declining sequence with respect to $(x, s, y)$, if the following conditions are satisfied.

(a) For every $i$, $1 \leqslant i \leqslant u$, $g_i$ is obtained from $g_{i-1}$ by a Coxeter transformation of kind $\neq$(C).

(b) The above conditions (i)–(iv) are satisfied, where $n = p(x, s, y)$ and $(x, s, y) = (x_0, s(0), y_0)$.

Thus the above discussion shows that

LEMMA 6. *For any* $(x, s, y) \in T$ *with* $p(x, s, y) > 0$, *there exists some declining sequence with respect to* $(x, s, y)$.

*Proof of Theorem* 3. Apply induction on $p(x, s, y) \geqslant 0$. If $p(x, s, y) = 0$ then it is just the assertion of Lemma 4. Now assume that $p(x, s, y) > 0$ and that the result has been shown for any $(x', s', y') \in T$ with $p(x', s', y') < p(x, s, y)$. By Lemma 6, there exists a declining sequence $g_0, g_1, ..., g_u$ with respect to $(x, s, y)$ described as above. Thus $p(x_u, s(u), y_u) < p(x, s, y)$. By the inductive hypothesis, we have $xsy = x_u s(u)\, y_u > x_u y_u = xy$. So our result is shown. ∎

Thus Theorem 1 follows as it is equivalent to Theorem 3.

By noting $l(xsy) \equiv l(xy)$ (mod $\cdot$ 2), we see that in Theorem 1 we have

$$l(xsy) \geqslant l(xy) + 1. \tag{1}$$

For any subset $J$ of $S$, let $W_J$ be the subgroup of $W$ generated by $J$.

THEOREM 7. *Given* $x, y \in W$, $J \subseteq S - \mathscr{R}(x)$, *and* $w \in W_J$, *if* $l(wy) =$

*$l(w)+l(y)$, then $xy \leqslant xwy$ and $l(xwy) \geqslant l(xy)+l(w)$. In particular, in the case when $l(xy)=l(x)+l(y)$, we have $l(xwy)=l(x)+l(y)+l(w)$.*

*Proof.* Apply induction on $l(w) \geqslant 0$. It is trivial when $l(w)=0$. If $l(w)=1$ then this is just the assertion of Theorem 1 by (1). Now assume that $l(w)>1$ and $s \in \mathscr{L}(w)$. Let $w'=sw$. Then $w' \in W_J$ and

$$l(w')+l(y) \geqslant l(w'y) \geqslant l(wy)-1=l(w)+l(y)-1=l(w')+l(y). \qquad (2)$$

This forces $l(w'y)=l(w')+l(y)$. So by the inductive hypothesis, we have $xy \leqslant xw'y$ and $l(xw'y) \geqslant l(xy)+l(w')$. From (2), we also see that $l(w'y)=l(wy)-1$ and so $s \notin \mathscr{L}(w'y)$. On the other hand, $s \in J \subseteq S-\mathscr{R}(x)$. Thus $s \notin \mathscr{L}(w'y) \cup \mathscr{R}(x)$. So by Theorem 1, we have $xy \leqslant xw'y < x \cdot s \cdot w'y = xwy$ and $l(xwy)=l(x \cdot s \cdot w'y) \geqslant l(xw'y)+1 \geqslant l(xy)+l(w')+1=l(xy)+l(w)$. This shows the first part of our result. Then the remaining part can be shown by the following inequalities

$$l(x)+l(y)+l(w) \geqslant l(xwy) \geqslant l(xy)+l(w)=l(x)+l(y)+l(w). \quad \blacksquare$$

## 2. Some Applications of Theorem 1

We shall make some applications of Theorem 1 in the present section. The first one is concerned with some multiplication properties of a Hecke algebra. Let $\mathscr{H}$ be the Hecke algebra over $A=\mathbb{Z}[u, u^{-1}]$ associated to a Coxeter group $(W, S)$ as below. $\mathscr{H}$ is a free $A$-module with a basis $\{T_w \mid w \in W\}$ and its multiplication satisfies the rules:

$$\begin{aligned} T_w T_{w'} &= T_{ww'}, && \text{if} \quad w, w' \in W \text{ with } l(ww')=l(w)+l(w'), \\ (T_s-u^{-1})(T_s+u) &= 0, && \text{for} \quad s \in S. \end{aligned} \qquad (1)$$

The rules (1) are equivalent to the rules

$$T_s T_w = \begin{cases} (u^{-1}-u)\, T_w + T_{sw}, & \text{if} \quad s \in \mathscr{L}(w), \\ T_{sw}, & \text{if} \quad s \notin \mathscr{L}(w). \end{cases} \qquad (2)$$

For any $x, y, z \in W$, define an element $f_{x,y,z} \in A$ by

$$T_x T_y = \sum_z f_{x,y,z} T_z. \qquad (3)$$

It is easily seen from (2) that $f_{x,y,z}$ is a polynomial in $v=u^{-1}-u$ of positive coefficients.

Define a subset

$$A(x, y)=\{z \in W \mid f_{x,y,z} \neq 0\}$$

of $W$ for any $x, y \in W$. It is well known that there exists a unique maximal element, written $\lambda(x, y)$, in $\Lambda(x, y)$ with respect to the Bruhat order $\leqslant$ (see [2]). That is,

$$\lambda(x, y) \geqslant z, \qquad \forall z \in \Lambda(x, y). \tag{4}$$

Here we shall apply Theorem 1 to show that there also exists a unique minimal element in $\Lambda(x, y)$ with respect to the same partial order.

THEOREM 8. *For any $x, y \in W$, we have $xy \in \Lambda(x, y)$ and*

$$xy \leqslant z, \qquad \forall z \in \Lambda(x, y). \tag{5}$$

*Proof.* Apply induction on $l(x) \geqslant 0$. It is trivial in case $l(x) = 0$. Now assume $l(x) > 0$. Let $s \in \mathscr{R}(x)$ and $x' = xs$. Then

$$T_x T_y = T_{x'} T_s T_y.$$

If $s \notin \mathscr{L}(y)$ then $T_x T_y = T_{x'} T_{sy}$ and so $\Lambda(x, y) = \Lambda(x', sy)$. Since $l(x') < l(x)$, this implies by the inductive hypothesis that

$$x' \cdot sy \leqslant z, \qquad \forall z \in \Lambda(x', sy).$$

So we get (5) in this case. If $s \in \mathscr{L}(y)$ then

$$T_x T_y = T_{x'}(v \cdot T_y + T_{sy}) = v \cdot T_{x'} T_y + T_{x'} T_{sy}.$$

By the positivity of the coefficients of the $f_{x,y,z}$'s, we have

$$\Lambda(x, y) = \Lambda(x', y) \cup \Lambda(x', sy).$$

By the inductive hypothesis and the fact that $l(x') < l(x)$, we get

$$x'y \leqslant z, \qquad \forall z \in \Lambda(x', y)$$

and

$$x'sy \leqslant z', \qquad \forall z' \in \Lambda(x', sy).$$

But $s \notin \mathscr{R}(x') \cup \mathscr{L}(sy)$. This implies from Theorem 1 that

$$xy = x'(sy) < x's(sy) = x'y.$$

Therefore we again get (5). ∎

Let $\deg f_{x,y,z}$ be the degree of the polynomial $f_{x,y,z}$ in $v = u^{-1} - u$. Here we state a property of $f_{x,y,z}$.

COROLLARY 9. *For any* $x, y \in W$ *and* $z \in A(x, y)$, *we have* $\deg f_{x,y,z} \geqslant 0$. *The equality holds if and only if* $z = xy$. *The constant term of* $f_{x,y,z}$ *in* $v$ *is equal to zero if* $z \neq xy$, *and is equal to* 1 *if* $z = xy$.

*Proof.* This follows directly from the multiplication rules of the Hecke algebra $\mathscr{H}$ and Theorem 8. ∎

The second application of Theorem 1 is to verify a conjecture of L. K. Jones.

Let $g$ be an expression of $W$. For any $s \in S$, let $n_s(g)$ be the number of the factor $s$ occurring in $g$. For any $w \in W$, let $I(w)$ be the set of all reduced expressions of $w$. Define a number

$$N_s(w) = \min\{n_s(g) \mid g \in I(w)\}.$$

Then Jones made the following conjecture which plays a crucial role in his paper.

*Conjecture* 10. Let $(W, S)$ be a symmetric group. Suppose that $x, y \in W$ and $s \in S$ satisfy the condition $s \notin \mathscr{L}(y) \cup \mathscr{R}(x)$. Then for any $t \in S - \{s\}$, we have $N_t(xy) \leqslant N_t(xsy)$.

To show the above conjecture, we shall prove the following result which includes this conjecture as a special case.

PROPOSITION 11. *Let* $(W, S)$ *be a Coxeter group. Let* $x, y \in W$ *and* $s \in S$ *be such that* $s \notin \mathscr{L}(y) \cup \mathscr{R}(x)$. *Then for any* $t \in S$, *we have* $N_t(xy) \leqslant N_t(xsy)$.

*Proof.* It is well known that for any reduced expression $s_1 s_2 \ldots s_r$ of $w$ with $s_i \in S$, there exists some subsequence $i_1, i_2, \ldots, i_t$ of $1, 2, \ldots, r$ such that $s_{i_1} s_{i_2} \ldots s_{i_t}$ is a reduced expression of $y$. This implies that

$$N_s(y) \leqslant N_s(w), \qquad \text{for any } y \leqslant w \text{ in } W \text{ and } s \in S.$$

So by Theorem 1, our result follows. ∎

## REFERENCES

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